SECTION 3.2: THE DERIVATIVE AS A FUNCTION

RECALL: One formula we have for the definition of derivative at $x = a$ is:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}, \quad \text{provided the limit exists}$$

Replacing 'a' with 'x', we get a formula for a derivative **function**:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided the limit exists}$$

EXAMPLE 1: Let $f(x) = x^2$.

- Find a formula for $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} &&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} &&= \lim_{h\to 0} (2x+h) \\
f'(x) &= 2x
\end{aligned}$$

- Use $f'(x)$ to write the equations of the tangent line at $x = -2$, $x = 0$, and $x = 1$.

  Check your answers geometrically using a graphing utility.

  - at $x = -2$: $f(-2) = (-2)^2 = 4$ and $f'(-2) = 2(-2) = -4$.

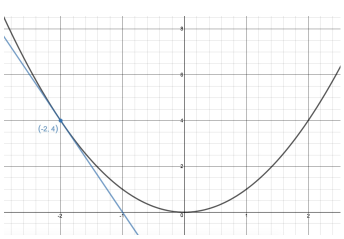
    The tangent line is: $y = f'(-2)(x - (-2)) + f(-2) = (-4)(x+2) + 4 = -4x - 4$, so $y = -4x - 4$.

  - at $x = 0$: $f(0) = (0)^2 = 0$ and $f'(0) = 2(0) = 0$.

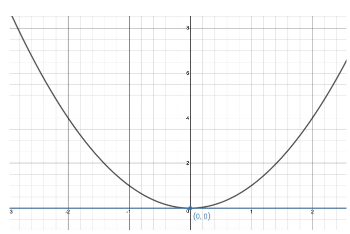
    The tangent line is: $y = f'(0)(x - (0)) + f(0) = (0)(x - (0)) + (0) = 0$, so $y = 0$.

  - at $x = 1$: $f(1) = (1)^2 = 1$ and $f'(1) = 2(1) = 2$.

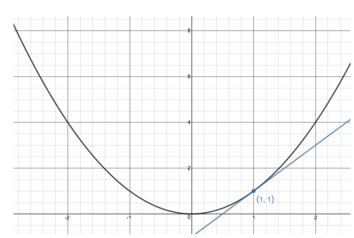
    The tangent line is: $y = f'(1)(x - (1)) + f(1) = (2)(x - 1) + 1 = 2x - 1$, so $y = 2x - 1$.



Tangent Line at $x = -2$



Tangent Line at $x = 0$



Tangent Line at $x = 1$

EXAMPLE 2: (VIDEO) Let $f(x) = \frac{1}{x}$.

- Find a formula for $f'(x)$. What values of x are excluded?

Ans: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \ldots = -\frac{1}{x^2},\ x \neq 0.$

- Use $f'(x)$ to write the equations of the tangent line at $x = -2$ and $x = 1$.

  Check your answers geometrically using a graphing utility.

Ans: at $x = -2$: $y = -\frac{1}{4}x - 1$; at $x = 1$: $y = -x + 2$

EXAMPLE 3: (VIDEO) The height, $s(t)$, in feet, of a model rocket t seconds after lift-off is given by:

$$s(t) = -5t^2 + 100t, \qquad 0 \le t \le 20$$

- Recall the **instantaneous velocity** of the rocket, $v(t) = s'(t)$. Find an expression for $v(t)$.

  Ans: $v(t) = s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = \lim_{h\to 0} \frac{[-5(t+h)^2 + 100(t+h)] - (-5t^2 + 100t)}{h} = \ldots = -10t + 100.$

- Find and interpret: $v(0)$, $v(5)$, and $v(15)$.

  $v(0) = 100$; at liftoff, the rocket is traveling **upwards** at a rate of 100 feet per second.

  $v(5) = 50$; 5 seconds after liftoff, the rocket is traveling **upwards** at a rate of 50 feet per second.

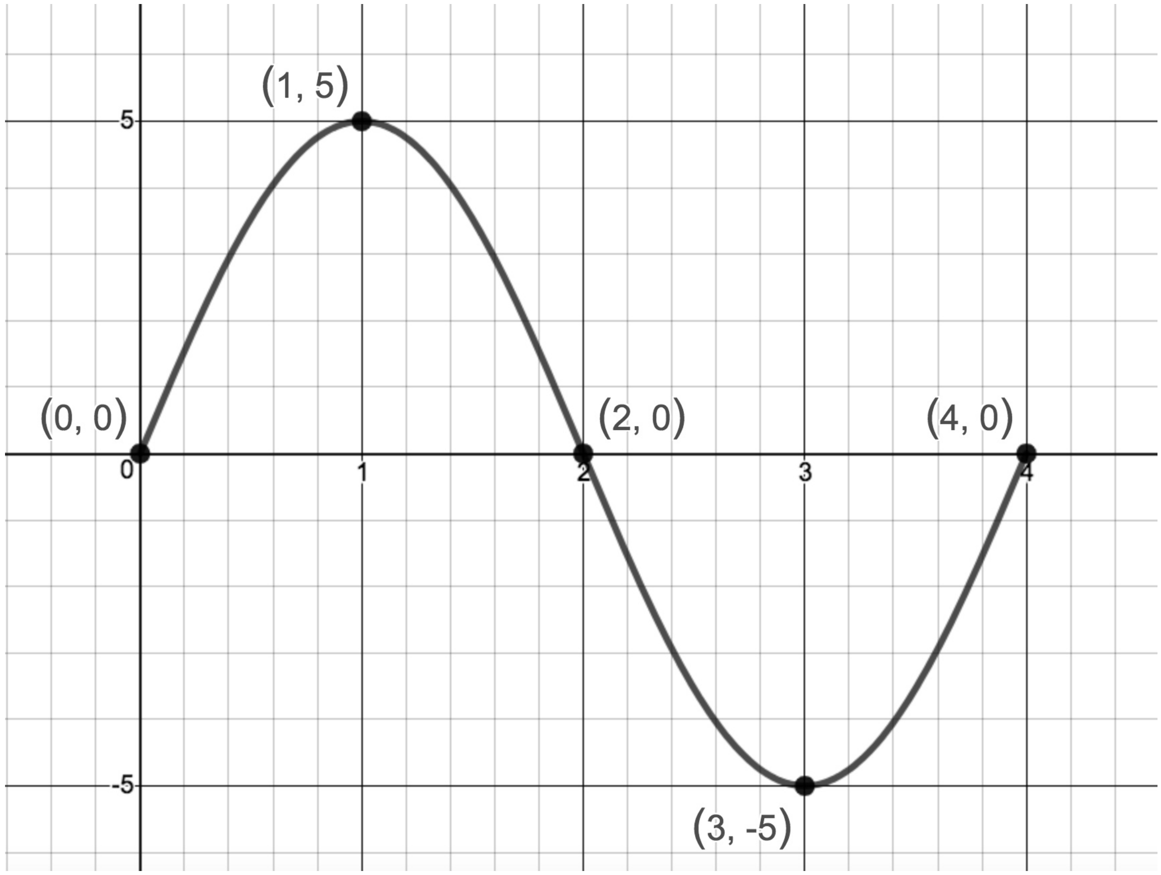
  $v(15) = -50$; 15 seconds after liftoff, the rocket is traveling **downwards** at a rate of 50 feet per second.

- Solve $v(t) = 0$. What does this mean in terms of the rocket?

  $v(t) = 0$ when $t = 10$ which means after 10 seconds, the rocket is still.

  At 10 seconds, the velocity changes from positive to negative which means the rocket momentarily pauses before changing direction from heading upwards to heading downwards.

EXAMPLE 4: Use the graph of $y = f(x)$ below to answer the following questions.



1. Is $f'(2)$ positive or negative or zero? How do you know?

   Ans: $f'(2) < 0$ (negative) since the slope of the tangent line is negative there.

2. Is $f'(3)$ positive or negative or zero? How do you know?

   Ans: $f'(3) = 0$ since the graph levels off there (horizontal tangent.)

3. Which is larger, $f'(0.4)$ or $f'(0.8)$? Explain.

   Ans: $f'(0.4) > f'(0.8)$: both slopes are positive and the tangent line at $x = 0.4$ is steeper than at $x = 0.8$

4. Use interval notation to describe the values of x for which:

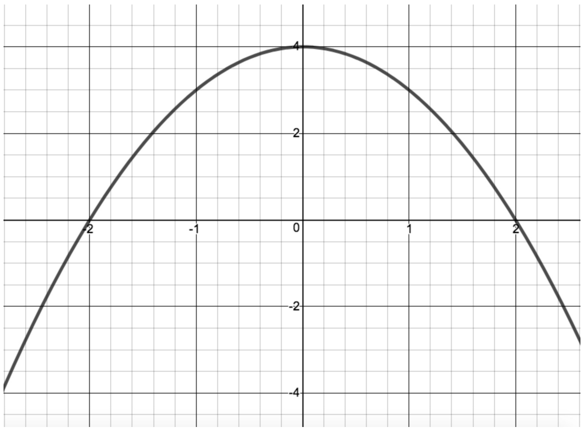
   $f'(x) < 0$:

   Ans: $f'(x) < 0$ on the interval $(1, 3)$ since the slopes of the tangent lines are all negative there.

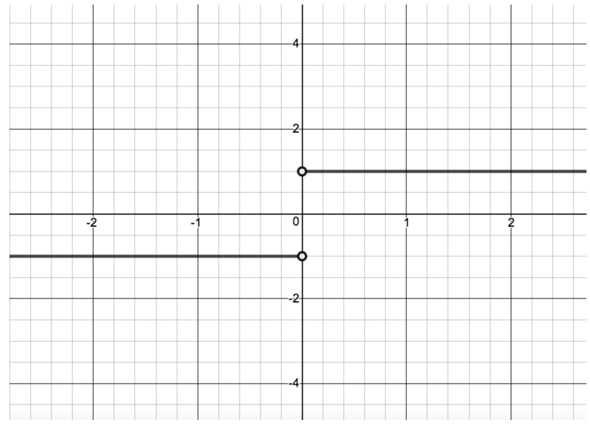
   $f'(x) > 0$:

   Ans: $f'(x) > 0$ on the intervals $(0, 1)$ and $(3, 4)$ since the slopes of the tangent lines are all positive there.

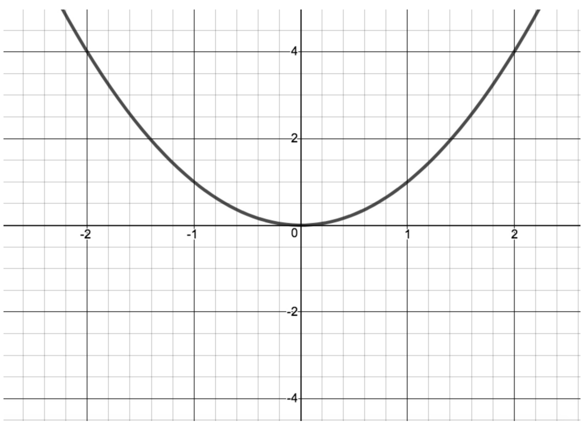
EXAMPLE 5: Match the graph of each function on the left with the graph of its derivative on the right.



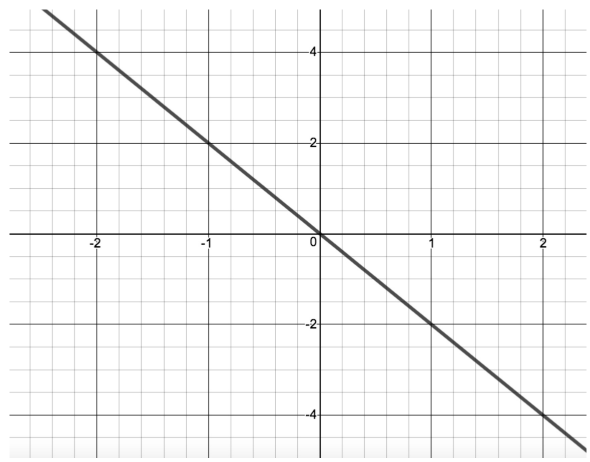
$y = f(x)$



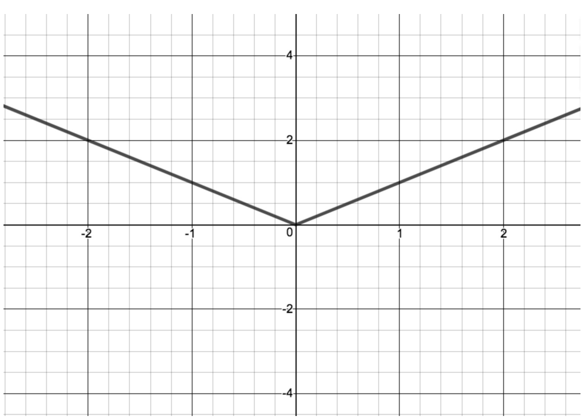
Graph A



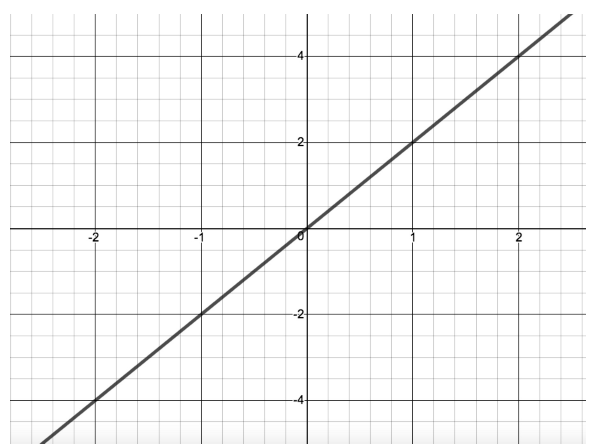
$y = g(x)$



Graph B



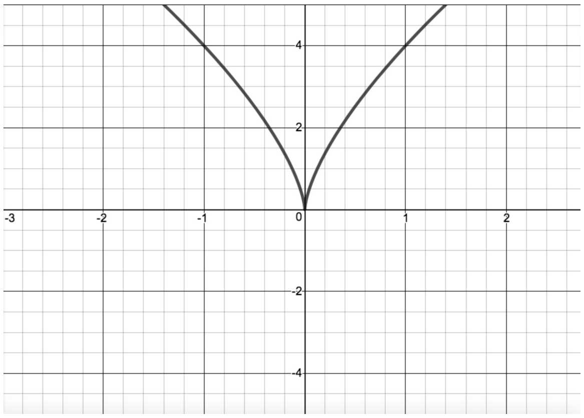
$y = h(x)$



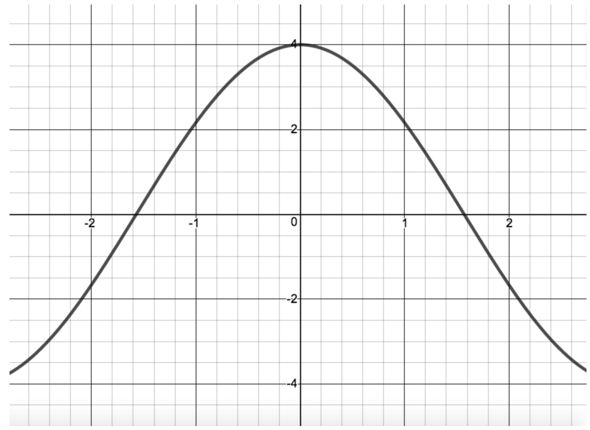
Graph C

Ans: $f'(x)$ is Graph B; $g'(x)$ is Graph C; $h'(x)$ is Graph A

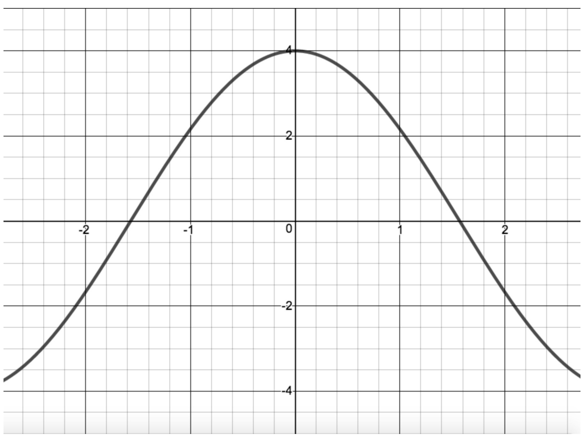
EXAMPLE 6: Match the graph of each function on the left with the graph of its derivative on the right.



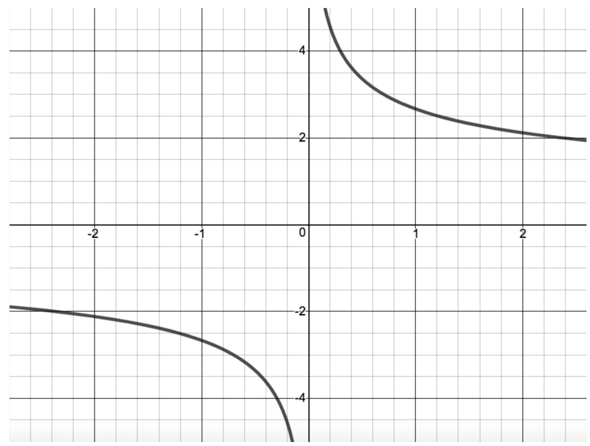
$y = f(x)$



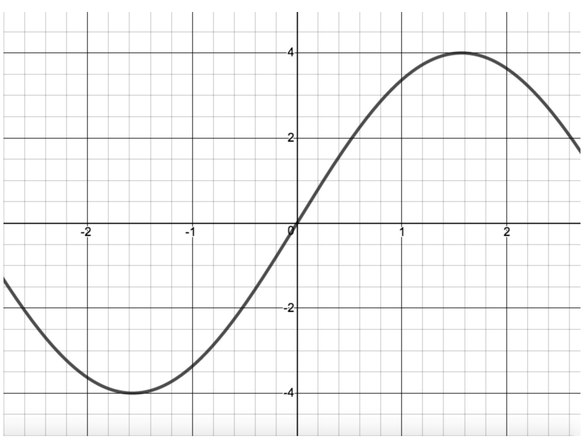
Graph A



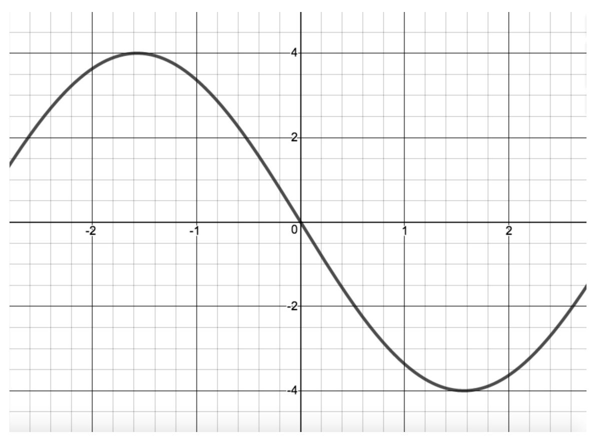
$y = g(x)$



Graph B



$y = h(x)$



Graph C

Ans: $f'(x)$ is Graph B; $g'(x)$ is Graph C; $h'(x)$ is Graph A

NOTE: It is helpful to keep in mind the difference between $f(x)$ and $f'(x)$:

- $f(x)$:
  - gives y-**value** on the graph of $y = f(x)$: $(x, f(x))$.
  - gives the value of the **output** of a function f.
- $f'(x)$:
  - gives the **slope of the tangent line** of the graph $y = f(x)$.
  - gives the **instantaneous rate of change** of a function f.

LOCAL LINEARITY

RECALL: A function f is called **differentiable** at $x = a$ if $f'(a)$ exists.
Let's delve a bit deeper into what this means.

Since $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$, if $f'(a)$ exists, then 'near' $x = a$, $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$.
Multiplying by $(x - a)$ gives $f'(a)(x - a) \approx f(x) - f(a)$ so solving for $f(x)$ gives

$$f(x) \approx f'(a)(x - a) + f(a) = \text{the output from the tangent line at } x = a$$

Geometrically, if we zoom in near $(a, f(a))$, the graph of $y = f(x)$ is indistinguishable from the tangent line.

In other words, if f is differentiable at $x = a$, then $f(x)$ is **locally linear** at $x = a$.

We make the '$\approx$' more precise in the following theorem whose proof is provided for the interested reader.

LOCAL LINEARITY THEOREM: (VIDEO) If f is differentiable at $x = a$, then there is a function $\epsilon(x)$ where:

$$f(x) = f'(a)(x - a) + f(a) + \epsilon(x)(x - a),$$

and $\epsilon(x) \to 0$ as $x \to a$.

PROOF: As already stated, f is differentiable at $x = a$ means $f'(a) \approx \dfrac{f(x) - f(a)}{x - a}$. Hence, we define:

$$\epsilon(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} - f'(a), & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases},$$

If $x \neq a$, then $\epsilon(x) = \dfrac{f(x) - f(a)}{x - a} - f'(a)$ so $\epsilon(x)(x - a) = f(x) - f(a) - f'(a)(x - a)$. Solving for $f(x)$, we get:

$$f(x) = f(a) + f'(a)(x - a) + \epsilon(x)(x - a).$$

In the case $x = a$, $\epsilon(x) = 0$ by definition, so the expression $f(a) + f'(x)(x - a) + \epsilon(x)(x - a)$ reduces to $f(a)$.

Hence, for all x, we have shown $f(x) = f(a) + f'(x)(x - a) + \epsilon(x)(x - a)$. Also note that:

$$\lim_{x\to a} \epsilon(x) = \lim_{x\to a} \left[\frac{f(x) - f(a)}{x - a} - f'(a) \right] = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} - \lim_{x\to a} f'(a) = f'(a) - f'(a) = 0,$$

so $\epsilon(x) \to 0$ as $x \to a$, as required.

Since differentiability means locally linear, it shouldn't be too surprising to learn the following:

DIFFERENTIABILITY IMPLIES CONTINUITY: If f is differentiable at $x = a$, then f is continuous at $x = a$.

'BIG IDEA' PROOF: If f is differentiable at $x = a$, then $f(x) \approx f'(a)(x-a) + f(a)$. Hence,

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f'(a)(x-a) + f(a)] = f'(a)(a-a) + f(a) = f(a)$$

Since $\lim\limits_{x\to a} f(x) = f(a)$, f is continuous at $x = a$.

FORMAL PROOF: Since f is differentiable at $x = a$, we use the Local Linearity Theorem to write: $f(x) = f'(a)(x-a) + f(a) + \epsilon(x)(x-a)$ for some function $\epsilon(x)$ where $\epsilon(x) \to 0$ as $x \to a$. Hence,

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f'(a)(x-a) + f(a) + \epsilon(x)(x-a)] = f'(a)(a-a) + f(a) + 0\cdot(a-a) = f(a).$$

Hence $\lim\limits_{x\to a} f(x) = f(a)$, so f is continuous at $x = a$.

NON-DIFFERENTIABLITY

EXAMPLE 7: Suppose f is **not** continuous at $x = a$. Is it possible for f to be differentiable at $x = a$? Explain.

Ans: No! Being differentiable at $x = a$ guarantees being continuous at $x = a$.

EXAMPLE 8: For each of the following functions:

- show f is continuous at $x = 0$.
- use the limit definition of derivative: $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$ to show f is **not** differentiable at $x = 0$.
- graph $y = f(x)$ near $x = 0$ using a graphing utility and zoom in near $(0, f(0))$ to verify your responses.

1. $f(x) = |x|$; **NOTE:** $f(x) = |x|$ has a **corner** at $(0,0)$.

   Ans: $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.

2. $f(x) = x^{\frac{2}{3}}$; **NOTE:** $f(x) = x^{\frac{2}{3}}$ has a **cusp** at $(0,0)$.

   Ans: $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^{2/3}}{x} = \lim\limits_{x\to 0}\dfrac{1}{x^{1/3}}$ does not exist.

3. $f(x) = \sqrt[3]{x}$; **NOTE:** $f(x) = \sqrt[3]{x}$ has a **vertical tangent line** at $(0,0)$.

   Ans: $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{\sqrt[3]{x}}{x} = \lim\limits_{x\to 0}\dfrac{x^{1/3}}{x} = \lim\limits_{x\to 0}\dfrac{1}{x^{2/3}}$ does not exist.

HOMEWORK: Section 3.2: 1 - 55 odd, 58*